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On the Summability of Lagrange Interpolation

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1. INTRODUCTION

Let

$$B \equiv \begin{matrix} x_1^{(1)} & & & \\ x_1^{(2)}, & x_2^{(2)} & & \\ \vdots & \vdots & & \\ x_1^{(n)}, & x_2^{(n)}, & \ldots, & x_n^{(n)} \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be an aggregate of points such that, for each n,

$$1 \geqslant x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqslant -1.$$

For any continuous function $f(x)$ with domain $[-1, 1]$ we define the nth Lagrange interpolation polynomial of $f(x)$ with respect to B to be that unique polynomial of degree at most $n - 1$ which assumes the values $f(x_1^{(n)}),\ldots, f(x_n^{(n)})$ at $x_1^{(n)}, x_2^{(n)},\ldots, x_n^{(n)}$, respectively.

Here we shall consider the case where the points $x_k^{(n)}$ $(k = 1, \ldots, n)$ are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$: that is

$$x_k^{(n)} = \cos(2k - 1)\,\pi/2n, \qquad k = 1,\ldots, n. \tag{1.1}$$

These interpolation polynomials are given by

$$L_n(f, x) = \sum_{k=1}^{n} f(x_k^{(n)})\, T_n(x)/((x - x_k^{(n)})\, T'(x_k^{(n)})).$$

Since, as it is easy to verify,

$$T_n(x)/(x - x_k^{(n)})\, T_n'(x_k^{(n)}) = (1/n)\left(1 + 2\sum_{r=1}^{n-1} T_r(x)\, T_r(x_k^{(n)})\right),$$

the $L_n(f)$ can be expressed as follows:

$$L_n(f, x) = \sum_{r=0}^{n-1} c_r(f)\, T_r(x), \tag{1.2}$$

where

$$c_0(f) = 1/n \sum_{k=1}^{n} f(x_k^{(n)}) \tag{1.3}$$

and

$$c_r(f) = 2/n \sum_{k=1}^{n} f(x_k^{(n)})\, T_r(x_k^{(n)}), \qquad r = 1,..., n - 1. \tag{1.4}$$

The properties of this sequence of polynomials are sometimes similar to those of the partial sums of the Fourier series of an integrable 2π-periodic function. Therefore, as in the theory of Fourier series, it is natural to consider summability methods which would sum the sequence $(L_n(f))$ to f for a large class of functions.

We shall consider summability methods

$$\Lambda_n(f, x) = \sum_{r=0}^{n-1} \lambda_r^{(n)} c_r(f)\, T_r(x), \tag{1.5}$$

which arise from a triangular matrix $(\lambda_k^{(n)})$ $k = 0, 1,..., n - 1$; $n = 1, 2,...$.

It is easy to see that

$$\Lambda_n(f, x) = \sum_{k=1}^{n} f(x_k^{(n)})\, A_{k,n}(x), \tag{1.6}$$

where

$$A_{k,n}(x) = (1/n)\left(1 + 2\sum_{r=1}^{n-1} \lambda_r^{(n)} T_r(x)\, T_r(x_k^{(n)})\right).$$

Here we prove the following theorem.

THEOREM. *Let the matrix of coefficients* $(\lambda_j^{(n)})$ *satisfy*

$$\lambda_0^{(n)} = 1; \qquad \lambda_j^{(n)} = 0 \quad \textit{if} \quad j \geq n; \quad \lambda_{n-1}^{(n)} = O(1/n) \tag{1.7}$$

and either

$$|\lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)}| = O(1/n^2), \qquad j = 1,..., n-1 \tag{1.8}$$

or

$$1 - \lambda_1^{(n)} = O(1/n), \qquad \lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)} \geqslant 0 \qquad j = 1,..., n-1. \tag{1.9}$$

Then

$$\| \Lambda_n(f) - f \| \leqslant c_1/n \sum_{r=1}^{n} \omega(1/r), \tag{1.10}$$

for $f \in C[-1, 1]$ *having modulus of continuity* $\omega(\delta)$. *The constant* c_1 *(and elsewhere* c_2, c_3,...) *is positive and independent of* n *and* f.

Now we give some choices of λ_j's which satisfy the above requirement.

$$\text{(a)} \quad \lambda_j^{(n)} = \frac{(n-j)^m}{(n-j)^m + j^m} \qquad j = 0, 1,..., n-1$$
$$= 0 \qquad j \geqslant n;$$

$$\text{(b)} \quad \lambda_j^{(n)} = \frac{(n-j)^m}{n^m} \qquad j = 0, 1,..., n$$
$$= 0 \qquad j \geqslant n.$$

2. Preliminaries

If $f(x) \equiv 1$ then

$$c_0(f) = 1$$

and, for $r = 1, 2,..., n-1$,

$$c_r(f) = 2/n \sum_{k=1}^{n} T_r(x_k^{(n)})$$
$$= 2/n \sum_{k=1}^{n} \cos(((2k-1)\, r/2n)\, \pi)$$
$$= 0.$$

So by (1.5) and (1.7)

$$\Lambda_n(1, x) \equiv 1,$$

and, therefore,

$$| A_n(f, x) - f(x)| \leqslant \sum_{k=1}^{n} | f(x_k^{(n)}) - f(x)| \, | A_{kn}(x)|$$

$$\leqslant \sum_{k=1}^{n} \omega(| x_k^{(n)} - x |) \, | A_{kn}(x)|.$$

Let $x = \cos\theta$, $x_k^{(n)} = \cos\theta_k^{(n)}$, $k = 1,..., n$. We have then

$$| A_n(f, x) - f(x)| \leqslant \sum_{k=1}^{n} \omega(| \theta_k^{(n)} - \theta |) \, | P_{k,n}(\theta)|, \tag{2.1}$$

where

$$\begin{aligned} P_{k,n}(\theta) &= A_{k,n}(\cos\theta) \\ &= (1/n) + (2/n) \sum_{r=1}^{n-1} \lambda_r^{(n)} \cos r\theta \cos r\theta_k^{(n)}. \end{aligned} \tag{2.2}$$

To prove the theorem we need some preliminary notation and estimates. We denote the Fejér kernel by

$$\begin{aligned} t_j(\theta) &= 1 + \frac{2}{j} \sum_{i=1}^{j-1} (j - i) \cos i\theta \\ &= \frac{1}{j} \left(\frac{\sin j\theta/2}{\sin\theta/2}\right)^2 \qquad \text{for} \quad j = 2, 3,..., n \end{aligned}$$

and $t_1(\theta) \equiv 1$.

Associated with this kernel we introduce

$$\tau_{j,k}(\theta) = \tfrac{1}{2}(t_j(\theta + \theta_k^{(n)}) + t_j(\theta - \theta_k^{(n)})).$$

It is easy to verify that

$$(j + 1)\, \tau_{j+1,k}(\theta) - 2j\tau_{j,k}(\theta) + (j - 1)\, \tau_{j-1,k}(\theta) = 2\cos j\theta \cos j\theta_k^{(n)}.$$

Using this relation we obtain

$$P_{k,n}(\theta) = 1/n \sum_{r=1}^{n-1} (\lambda_{r-1}^{(n)} - 2\lambda_r^{(n)} + \lambda_{r+1}^{(n)})\, r\tau_{r,k}(\theta) + \lambda_{n-1}^{(n)} \tau_{n,k}(\theta). \tag{2.3}$$

If there is no confusion we shall write A_k, P_k, λ_k, θ_k, for $A_{k,n}$, $P_{k,n}$, $\lambda_k^{(n)}$, $\theta_k^{(n)}$.

Naturally enough, we shall require the following lemma.

LEMMA 1. *Under the hypotheses* (1.7), (1.8) *or* (1.7), (1.9) *we have*

$$\sum_{k=1}^{n} | A_k(x)| = \sum_{k=1}^{n} | P_k(\theta)| = O(1).$$

Proof. Let (1.7) and (1.8) hold. That is,

$$\lambda_0 = 1, \qquad \lambda_j = 0 \qquad \text{if} \quad j \geqslant n, \qquad \lambda_{n-1} = O(1/n)$$

and

$$| \lambda_{j+1} - 2\lambda_j + \lambda_{j-1} | = O(1/n^2) \qquad j = 1,..., n-1.$$

Then by these hypotheses and (2.3) we have

$$\sum_{k=1}^{n} | P_k(\theta)| \leqslant \sum_{k=1}^{n} 1/n \left(\sum_{j=1}^{n-1} | \lambda_{j+1} - 2\lambda j + \lambda_{j-1} | j\tau_{j,k}(\theta)\right) + \sum_{k=1}^{n} | \lambda_{n-1} | \tau_{n,k}(\theta)$$

$$= 1/n \sum_{j=1}^{n-1} O(1/n^2), jn + O(1/n)n$$

$$= O(1).$$

Alternatively, let (1.7) and (1.9) hold. That is

$$\lambda_0 = 1, \qquad \lambda_j = 0 \qquad \text{if} \quad j \geqslant n, \qquad \lambda_{n-1} = O(1/n)$$

and

$$1 - \lambda_1 = O(1/n), \qquad \lambda_{j+1} - 2\lambda j + \lambda_{j-1} \geqslant 0, \qquad j = 1,..., n-1.$$

Then we have

$$\sum_{k=1}^{n} | P_k(\theta)| \leqslant \sum_{j=1}^{n-1} (\lambda_{j+1} - 2\lambda_j + \lambda_{j-1})\, j + n0(1/n) = O(1).$$

LEMMA 2. *Let* $\theta \neq \theta_k$. *Then for* $1 \leqslant k \leqslant n$ *and* $1 \leqslant r \leqslant n$,

$$\tau_{r,k}(\theta) \leqslant \pi^2/r(\theta - \theta_k)^2.$$

Proof. By definition,

$$\tau_{r,k}(\theta) = \frac{1}{2r}\left(\frac{\sin^2(r(\theta + \theta_k))/2}{\sin(\theta + \theta_k)/2} + \frac{\sin^2(r(\theta - \theta_k))/2}{\sin^2(\theta - \theta_k)/2}\right). \tag{2.4}$$

Also

$$\sin\frac{\theta + \theta_k}{2} = \sin\frac{\theta}{2}\cos\frac{\theta_k}{2} + \cos\frac{\theta}{2}\sin\frac{\theta_k}{2},$$

and, hence,

$$\left| \sin \frac{\theta + \theta_k}{2} \right| \geqslant \left| \sin \frac{\theta}{2} \sin \frac{\theta_k}{2} - \cos \frac{\theta}{2} \sin \frac{\theta_k}{2} \right| = \sin \left| \frac{\theta - \theta_k}{2} \right|. \quad (2.5)$$

Then the lemma follows from (2.4), (2.5), and Jordan's inequality, namely,

$$|\sin x| \geqslant 2/\pi \, |x| \qquad \text{if} \quad 0 \leqslant |x| \leqslant \pi/2.$$

LEMMA 3. *Let $\theta \in [0, \pi]$, and let $\theta_j = ((2j-1)\pi)/2n$ be the node nearest to θ. Then*

$$\sum_{k=1}^{j-1} (\omega(|\theta_k - \theta|))/(\theta_k - \theta)^2 \leqslant c_2 n \sum_{r=1}^{n} \omega(1/r)$$

and

$$\sum_{k=j+1}^{n} (\omega(|\theta_k - \theta|))/(\theta_k - \theta)^2 \leqslant c_3 n \sum_{r=1}^{n} \omega(1/r).$$

(If $j = 1$ or n then only one of these inequalities holds.)

Proof. This lemma is contained implicitly in a paper given by Bojanic [1].

3. PROOF OF THE THEOREM

We can now prove the theorem. Let j be as in Lemma 3. By (2.1),

$$|A_n(f, x) - f(x)| \leqslant \sum_{k=1}^{j-1} \omega(|\theta_k - \theta|) \, |P_k(\theta)| + \omega(|\theta_j - \theta|) \, |P_j(\theta)|$$

$$+ \sum_{k=j+1}^{n} \omega(|\theta_k - \theta) \, |P_k(\theta)|.$$

As remarked before, the first or last sum may not appear in some cases. By our choice of θ_j and Lemma 1,

$$\begin{aligned} \omega(|\theta_j - \theta|) \, |P_j(\theta)| &\leqslant c_3 \omega(1/n) \, |P_j(\theta)| \\ &\leqslant c_4 \omega(1/n) \\ &\leqslant C_5/n \sum_{r=1}^{n} \omega(1/r). \end{aligned} \quad (2.9)$$

Suppose (1.7) and (1.8) hold. Then by these hypotheses and Lemmas 2 and 3,

$$\sum_{k=1}^{j-1} \omega(|\theta_k - \theta|)\,|P_k(\theta)|$$

$$\leqslant \sum_{k=1}^{j-1} \omega(|\theta_k - \theta|)\left(1/n \sum_{r=1}^{n-1} |\lambda_{r+1} - 2\lambda_r + \lambda_{r-1}|\, r\tau_{r,k}(\theta) + |\lambda_{n-1}|\,\tau_{n,k}(\theta)\right)$$

$$\leqslant c_6/n^3 \sum_{k=1}^{n-1} r \sum_{k=1}^{j-1} \omega(|\theta_k - \theta|)\,\tau_{r,k}(\theta) + \sum_{k=1}^{j-1} |\lambda_{n-1}|\,\omega(|\theta_k - \theta|)\,\tau_{n,k}(\theta)$$

$$\leqslant c_7/n^3 \sum_{r=1}^{n-1} \sum_{k=1}^{j-1} (\omega(|\theta_k - \theta|)/(\theta_k - \theta)^2) + c_8/n \sum_{k=1}^{j-1} \omega(|\theta_k - \theta|)\,\tau_{r,k}(\theta)$$

$$\leqslant c_9/n \sum_{r=1}^{n} \omega(1/r). \tag{2.10}$$

A similar estimate is valid for $\sum_{k=j+1}^{n} \omega(|\theta_k - \theta|)\,|P_k(\theta)|$ and so by (2.9) and (2.10) the proof is complete.

It remains to consider the case when (1.7) and (1.9) hold. Since the inequality (2.9) is still valid, it suffices to estimate the sum $\sum_{k=1}^{j-1}$ under these conditions.

Now

$$A \equiv \sum_{k=1}^{j-1} \omega(|\theta - \theta_k|)\,|P_k(\theta)|$$

$$\leqslant \sum_{k=1}^{j-1} \omega(|\theta - \theta_k|)\left(1/n \sum_{r=1}^{n-1} (\lambda_{r+1} - 2\lambda_r + \lambda_{r-1})\, r\tau_{r,k}(\theta) + |\lambda_{n-1}|\,\tau_{n,k}(\theta)\right)$$

$$= 1/n \sum_{r=1}^{n-1} (\lambda_{r+1} - 2\lambda_r + \lambda_{r-1})\, r \sum_{k=1}^{j-1} \omega(|\theta - \theta_k|)\,\tau_{r,k}$$

$$+ |\lambda_{n-1}| \sum_{k=1}^{j-1} \omega(|\theta - \theta|)\,\tau_{n,k}(\theta).$$

In using Lemmas 2 and 3 and the hypotheses (1.7) and (1.9).

$$A \leqslant 1/n \sum_{r=1}^{n-1} (\lambda_{r+1} - 2\lambda_r + \lambda_{r-1}) \sum_{k=1}^{j-1} (\omega(|\theta - \theta_k|)/(\theta - \theta_k)^2)$$

$$+ c_{10}/n^2 \sum_{k=1}^{j-1} (\omega(|\theta - \theta_k|)/(\theta - \theta_k)^2)$$

$$\leqslant c_{11} \left(\sum_{r=1}^{n} \omega(1/r)\right) (1 - \lambda_1 - \lambda_{n-1}) \leqslant c_{12}/n \sum_{r=1}^{n} \omega(1/r).$$

Again a similar estimate is valid for $\sum_{k=j+1}^{n}$ and the proof of the theorem is complete.

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References

1. R. Bojanic, A note on the precision of interpolation by Hermite-Fejér polynomials. "Proceedings of the Conference on the Constructive Theory of Functions held in Budapest, 1969," pp. 69–76, Budapest, 1972.